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An Association Scheme for the 1-Factors of the Complete Graph

B. M. I. RANDS

Department of Mathematics,
California Institute of Technology, Pasadena, California 91125

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INTRODUCTION

Johnson and Hamming schemes have been put to good use by Delsarte [2] and others in the theory of designs and error-correcting codes, and it is hoped that the same ideas can be applied to collections of 1-factors by making use of the association scheme herein described. The 1-factors of the complete graph on $2n$ vertices are shown to form an association scheme with $p(n)-1$ classes; a general method for calculating the eigenvalues of the scheme, using group representation theory, is given. These eigenvalues are computed for the cases $n=4, 5, 6$.

DEFINITIONS. A 1-factor of K_{2n} is a collection of n edges of the complete graph on $2n$ vertices, such that any vertex lies on a unique edge. A given 1-factor can be identified with the fixed point free involution of S_{2n} which interchanges the pairs of points that make up the edges of the 1-factor. The natural action of the symmetric group S_{2n} on the 1-factors of K_{2n} is permutation equivalent to its action by conjugation on its fixed point free involutions.

An association scheme on a set Ω of size v is a partition of the 2-element subsets of Ω into m classes or relations $R_1, \ldots, R_m$ satisfying

$$x \in \Omega \Rightarrow |\{y: \{y, x\} \in R_i\}| = v_i \tag{1}$$

$$x \neq y \in \Omega, \{x, y\} \in R_i \Rightarrow |\{z: \{z, x\} \in R_j, \{z, y\} \in R_k\}| = p_{ijk} \tag{2}$$

for nonnegative integers v_i $(i=1, \ldots, m)$ and p_{ijk} $(i, j, k = 1, \ldots, m)$.

To each relation R_i we can associate a graph on Ω with $v \times v$ adjacency matrix, A_i, and if $A_0 = I$, the identity matrix, we have $J = A_0 + \cdots + A_m$, where J is the $v \times v$ matrix of all 1's. Furthermore, we have $A_k A_j = A_j A_k =$

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$\sum_{i=0}^{m} p_{ijk} A_i$, where the p_{ijk}'s with one or more subscripts zero, are suitably defined. So the A_i's span an associative and commutative algebra $\mathcal{Cl}$.

Let $V = C^v$ be the vector space on which the matrices A_i can be said to act. Then there exists an orthogonal decomposition $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ (the eigenspaces of the A_i's, which are simultaneously diagonalisable) such that if E_i denotes the matrix of the orthogonal projection $V \to V_i$, then

$$\mathcal{Cl} = \operatorname{span}\{A_i\}_{i=0}^{m} = \operatorname{span}\{E_i\}_{i=0}^{m}.$$

It is customary to take $V_0 = \operatorname{span}\{(1, 1,..., 1)\}$. The V_i's are known as the eigenspaces of the algebra, and dim $V_i = \mu_i$, say.

We have, $A_j = P_j(0) E_0 + P_j(1) E_1 + \cdots + P_j(m) E_m$ for scalars $P_j(k)$, which are known as the eigenvalues of the scheme. It is not difficult to show that the $P_j(k)$'s determine all the parameters of the scheme.

Delsarte develops conditions on the distribution vector of a subset of elements of the scheme in terms of the eigenvalues. Given a subset $Y \subseteq \Omega$, its distribution vector $\mathbf{a}$ is an $(m + 1)$-dimensional vector with

$$a_0 = 1, \qquad a_i = (1/|Y|) \sum_{y,z \in Y} |\{\{y, z\}\} \cap R_i|, \qquad i = 1,..., m.$$

Delsarte's inequalities state that

$$\sum_{j=0}^{m} a_j P_i(j)/v_j \geqslant 0 \qquad \text{for} \quad i = 0,..., m.$$

and so a knowledge of the eigenvalues restricts the possible subsets of the scheme.

The centralizer algebra of a transitive permutation group G on a set Ω of size v, is the set of $v \times v$ matrices over C which commute with all the permutation matrices of G. If the group G has rank $n + 1$ on Ω, and the centralizer algebra is commutative, then the orbits of G on the 2 subsets of Ω form an n class association scheme. The definition of the centralizer algebra and Theorems A and B can be found in Wielandt [6].

THEOREM A. *The centralizer algebra of G on Ω is commutative if and only if the permutation character of G on Ω is the sum of distinct irreducible characters of G.*

For each $g \in G$, let $P(g)$ be the $v \times v$ permutation matrix defined by the action of g on Ω. For the ith conjugacy class $\mathscr{C}_i$ of G, define the ith class matrix

$$C_i = \sum_{g \in \mathscr{C}_i} P(g).$$

Then we have

THEOREM B. *All the class matrices belong to the centralizer algebra V. They commute with each other. Furthermore, the centralizer algebra is commutative if and only if the class matrices generate V.*

Finally, we make use of a theorem proved by Thompson [5], and also by Saxl and James, see [4], but perhaps known earlier, concerning the representation of the symmetric group, S_{2n} acting on its fixed point free involutions by conjugation.

THEOREM C. *Let χ be the character of this permutation representation and let $\chi^{(\pi)}$ be the irreducible character of S_{2n} corresponding to the partition π. Then,*

$$(\chi, \chi^{(\pi)}) = 1, \qquad \text{if every part of } \pi \text{ is even,}$$
$$= 0, \qquad \text{otherwise.}$$

In particular, this representation has rank $p(n)$, the number of partitions of n.

Theorems A and C together imply that the 1-factors of K_{2n} give rise to a scheme with $p(n) - 1$ classes. The main point of Theorem C is that it identifies the eigenspaces of the association scheme with certain irreducible representation modules of S_{2n}. We give a proof that the rank of the representation is $p(n)$ which will also be of use later.

LEMMA. *Let f_1, f_2, f_3 be 1-factors of K_{2n}.*

(1) *For every partition π of n there corresponds a relation between the 1 factors.*

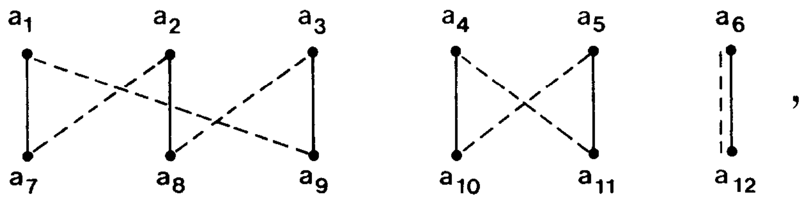
(2) *If $\{f_1, f_2\}$, and $\{f_1, f_3\} \in R(\pi)$, then there exists $\sigma \in S_{2n}$ such that $f_1\sigma = f_1$ and $f_2\sigma = f_3$.*

(3) *There exists $\sigma \in S_{2n}$ such that $f_1\sigma = f_2$ and $f_2\sigma = f_1$.*

(4) *There exists $\sigma \in S_{2n}$ such that σ fixes at least n of the $2n$ points and $f_1\sigma = f_2$.*

In fact, if $f_1 + f_2$ (for definition see below) has x components of length two, and y components of length greater than two, then there are exactly 2^y permutations $\sigma \in S_{2n}$ which fix $n + x$ points and such that $f_1\sigma = f_2$.

Proof. (1) Take any two 1-factors f_1, f_2. Their edges taken together, i.e., $f_1 + f_2$, cover every vertex twice and so form a collection of disjoint circuits of even length (including length 2, ) and so we obtain a partition π of $2n$ into even parts, and so $\frac{1}{2}\pi$ is a partition of n. Conversely, given a partition π of n, 2π is a partition of $2n$, and it is not difficult to find 1-factors f_1, f_2 such that the circuits of $f_1 + f_2$ have lengths the constituents of 2π.



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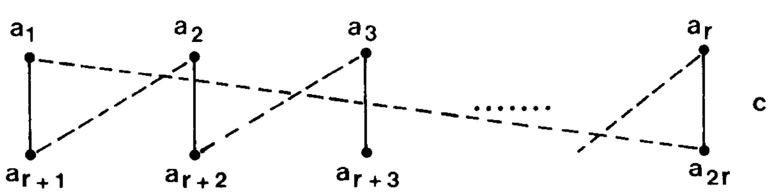
(2) Suppose $f_1 + f_2$ and $f_1 + f_3$ have as components circuits of the same lengths, i.e., they both correspond to the same partition of n. Consider first, $f_1 + f_2$, and the longest circuit c_1, of $f_1 + f_2$, of length $2m_1$. Then, any point in this circuit lies on one edge of f_1 and one edge of f_2. Pick an arbitrary point and label it a_1. Then go along the f_1 edge at a_1, and label the adjacent point a_{n+1}. Now go along the f_2 edge at a_{n+1} and label the next point a_2. Continue in this manner until all the points of c_1 have been labelled, with labels $a_1,\dots, a_{m_1}$, $a_{n+1},\dots, a_{n+m_1}$. Now label the other circuits similarly as indicated in Insert 1 for $n = 6$, where f_1 edges are continuous and f_2 edges are dotted. Now, produce a similar labelling for $f_1 + f_3$ with labels $b_1,\dots, b_{2n}$. Then, because $f_1 + f_2$ and $f_1 + f_3$ correspond to the same partition of n, $\sigma \in S_{2n}$ such that $a_i \to b_i$ ($i = 1$ to $2n$) fixes f_1 and maps f_2 to f_3.

(3) For each circuit c_i of $f_1 + f_2$ choose an orientation $\gamma(c_i)$ of the points of c_i. Then, $\sigma = \prod \gamma(c_i)$ is such that $f_1\sigma = f_2$ and $f_2\sigma = f_1$.

(4) In the diagram $a_1 \to a_2 \to a_3 \to a_1$, $a_4 \to a_5 \to a_4$ is such a σ.

In fact, if we let $\mathrm{p}(f_i)$ be the fixed point free involution corresponding to f_i, $i = 1, 2$, then $\mathrm{p}(f_1)\,\mathrm{p}(f_2) = \sigma\sigma'$, where σ and σ' fix at least n points and $f_1\sigma = f_2$. (See Thompson's proof of Theorem C.)

To prove the last part of (4), we consider a particular circuit c of $f_1 + f_2$ of length $2r$ greater than two. We again perform the labelling in Insert 2. Then $(a_1, a,\dots, a_r)$ and $(a_{2r} a_{2r-1} \cdots a_{r+1})$ are two such $\sigma \in S_{2n}$ which fix at least r points of c and map the edges of f_1 to the edges of f_2. Now, clearly, there can be no $\sigma \in S_{2n}$ which maps the edges of f_1 to the edges of f_2, and fixes two consecutive points of c. So any such σ fixes at most r points of c (and no two



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consecutive ones), and so any σ which fixes exactly r points of c, fixes alternate points. So any edge of f has one of its points fixed, and so the other point in the edge must go either to the point two to the left in an orientation of c, or to a point two to the right. This determines the action of σ on the circuit. Finally, the only way we can have σ mapping f_1 to f_2 and fixing $n+x$ points, is for it to fix each repeated edge and act in one of two ways on each circuit of length greater than two. Hence the result. Conditions (1)–(3) are enough to show that we have an association scheeme with p(n) classes. Condition (4) will be of use in the next section.

The Eigenvalues of the Scheme

For the Johnson and Hamming schemes, the eigenvalues are known in terms of the basic parameters and Krawchuk or Eberlein polynomials. It does not seem possible to provide such a neat closed form for the eigenvalues of our scheme, but we present a method for finding them, making use of Theorem B.

Clearly, the class matrices here are not independent and there is no unique way of representing the A_i's as combinations of the C_j's. A canonical expression can be found, however, in terms of the C_j's for only those conjugacy classes whose elements fix at least n points, and in terms of these C_j's the expression is unique. The reason for this is part (4) of the preceeding lemma.

Now, the number of conjugacy classes whose elements fix at least n points is exactly p(n), and it turns out that we have a triangular system of equations connecting the C_j's and the A_i's. First, we index the A_i's, C_j's, and E_k's by partitions of n as follows: Let π be a partition of n. Then, $A(\pi)$ is the adjacency matrix corresponding to the relation $R(\pi)$ such that f_1, $f_2 \in R(\pi) \Leftrightarrow$ the circuits of $f_1 + f_2$ have as lengths the components of 2π. Then, $C(\pi)$ is the class matrix corresponding to the conjugacy class in S_{2n} whose cycle lengths are components of π together with n fixed points. Matrix $E(\pi)$ is the projection matrix of V onto $V(\pi)$, the eigenspace which is isomorphic as an S_{2n} module to the Specht module arising from the partition $2(\pi')$ of $2n$, where π' is the conjugate partition to π. The ordering of the partitions π of n follows that used by Littlewood [3] in his character tables of the symmetric groups.

The eigenvalues of the C_i's on the irreducible subspaces are well known, and easily calculated from the character table of S_{2n}, or if this is not available, by the methods of the representation theory of the symmetric group. We have

$$\lambda_k(C_i) = |\mathscr{C}_i|\, \chi^k(i)/\chi^k(1).$$

The main problem is to find a_{ij}'s such that $C_i = \sum_{j=1}^{i} a_{ij} A_j$. This is done for the cases $n = 4, 5, 6$, but beyond this point the calculations become very involved.

We note, in passing, that this problem is equivalent to determining the composition, in terms of conjugacy classes, of each coset of the stabiliser of a 1-factor and is not a problem where the existing group theory is much help.

Computing the a_{ij}'s

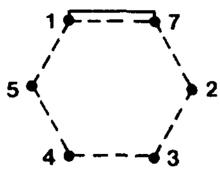
We recall $C_i(f_1, f_2)$ is equal to the number of $\sigma \in \mathscr{C}_i$ such that $f_1 \sigma = f_2$. So a_{ij} is equal to the number of $\sigma \in \mathscr{C}_i$ such that if $\{f_1, f_2\} \in R_j$, then $f_1 \sigma = f_2$. For small n, and for some i and j, this is a relatively satisfactory method for calculating the a_{ij}'s. The length of these computations, however, increases rapidly with n, and for $n = 5$, 6 the following shorter method was used:

The underlying idea is to pick a particular f_1 and count the number of $\sigma \in \mathscr{C}_i$ such that $\{f_1, f_1\sigma\} \in R_j$. Calling this number b_{ij}, we have $a_{ij} = b_{ij}/v_j$.

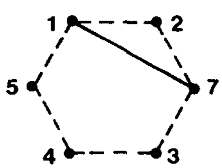
The calculation of the b_{ij}'s falls into two parts. First, we find the number of $\sigma \in \mathscr{C}_i$ which contain certain edges of f_1, in their cycle structure in a prescribed manner, and then we find the number of ways we can choose such edges. The best way to illustrate the method is by a example. We do the case, where

$$n = 6, \qquad \mathscr{C}_i = \{(123456)\},$$

$$f_1 = (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12).$$

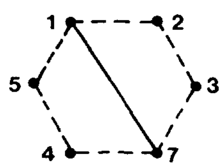


48σ such that $\{f_1, f_1\sigma\} \in R_6$,



$48R_6$

giving $$a_{7,6} = \frac{120 \times 6 \times 5 \times 2^4}{2304} = 25,$$

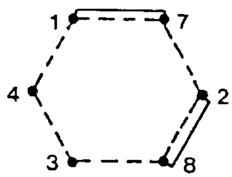


$24R_6$.

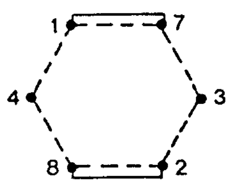
INSERT 3

First, there is the case where the 6 cycle consists of one point from each edge of the 1-factor. As we have remarked in part (4) of the lemma, this gives rise to $a_{ii}=2$. Second, there is the case where the 6 cycle contains 2 points from 1 edge of the 1-factor and 4 other points, one from each of four edges of the 1-factor. Then, we have three subcases (Insert 3).

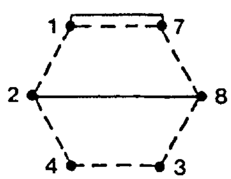
Third, there is the case where the 6 cycle contains 2 edges of the 1-factor and 2 other points, one from each of two edges of the 1-factor. Then we have 8 subcases (Insert 4).



$16R_3$



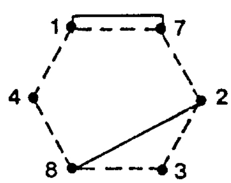
$8R_3$



$16R_3$

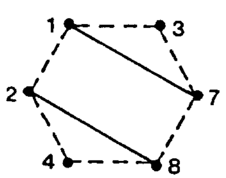
These contribute

$$\frac{80\times 15\times 6\times 2^2}{720}=40A_3,$$



$32R_3$

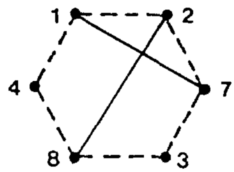
$$\frac{16\times 15\times 6\times 2^2}{180}=32A_4,$$



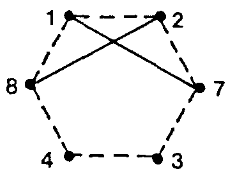
$8R_3$

$$\frac{24\times 15\times 6\times 2^2}{160}=54A_2.$$

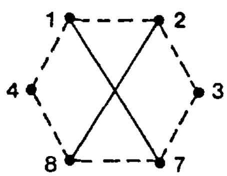
INSERT 4



$16R_4$



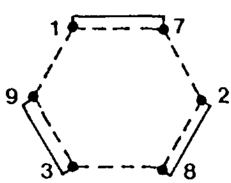
$16R_2$



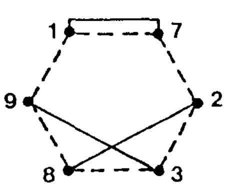
$8R_2$.

INSERT 4—*Continued*

Fourth, there is the case, where the 6 cycle contains 3 edges of the 1-factor We then have 5 subcases (Insert 5).

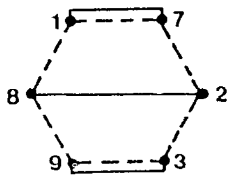


$16R_2$



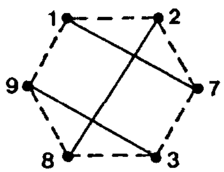
$48R_1$

INSERT 5



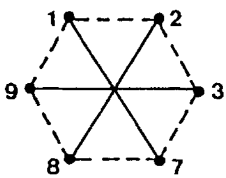
$24R_2$ These contribute

$$\frac{48 \times 20}{30} = 32A_1,$$



$24R_2$

$$\frac{64 \times 20}{160} = 8A_2,$$



$8R_0$ $\quad 8 \times 20 = 160A_0.$

INSERT 5—*Continued*

From all this we obtain $C_7 = 160A_0 + 32A_1 + 62A_2 + 40A_3 + 32A_4 + 25A_6 + 2A_7$.

The Eigenvalues and the a_{ij}'s

Case $n = 4$. p(4) = 5. See Table I.

(1) $\quad C_0 = A_0 = I$;

(12) $\quad C_1 = 4A_0 + 2A_1$;

(123) $\quad C_2 = 4A_1 + 2A_2$;

(1234) $\quad C_3 = 12A_0 + 2A_1 + 9A_2 + 2A_3$;

(12)(34) $\quad C_4 = 18A_0 + 4A_1 + 3A_2 + 4A_4$.

TABLE I

P matrix	1^4	$2, 1^2$	$3, 1$	4	$2, 2$	μ_i
[8]	1	12	32	48	12	1
[6, 2]	1	5	−4	8	−2	20
$[4^2]$	1	2	8	−2	7	14
$[4, 2^2]$	1	−1	−2	4	−2	56
$[2^4]$	1	−6	8	−6	3	14

Case $n = 5$. p(5) = 7. See Table II.

(1)	$C_0 = A_0;$
(12)	$C_1 = 5A_0 + 2A_1,$
(123)	$C_2 = 4A_1 + 2A_2,$
(1234)	$C_3 = 20A_0 + 2A_1 + 9A_2 + 2A_3,$
(12)(34)	$C_4 = 30A_0 + 6A_1 + 3A_2 + 4A_4;$
(12)(345)	$C_5 = 36A_1 + 10A_2 + 8A_3 + 16A_4 + 4A_5;$
(12345)	$C_6 = 24A_1 + 12A_2 + 16A_2 + 2A_6.$

TABLE II

P matrix	1^5	$2, 1^3$	$3, 1^2$	4, 1	$2^2, 1$	2,	5	μ_i
[10]	1	20	80	240	60	160	384	1
[8, 2]	1	11	26	24	6	−20	−48	35
[6, 4]	1	6	−4	−26	11	20	−8	90
$[6, 2^2]$	1	3	2	−8	−10	−4	16	225
$[4^2, 2]$	1	0	−10	10	5	10	4	252
$[4, 2^3]$	1	−4	2	6	−3	10	−12	300
$[2^5]$	1	−10	20	−30	15	−20	24	42

Case $n = 6$. p(6) = 11. See Table III.

(1)	$C_0 = A_0;$
(12)	$C_1 = 6A_0 + 2A_1;$
(123)	$C_2 = 4A_1 + 2A_2;$
(1234)	$C_3 = 30A_0 + 2A_1 + 9A_2 + 2A_3;$
(12)(34)	$C_4 = 45A_0 + 8A_1 + 3A_2 + 4A_4;$
(12)(345)	$C_5 = 48A_1 + 12A_2 + 8A_3 + 16A_4 + 4A_5,$
(12345)	$C_6 = 32A_1 + 12A_2 + 16A_3 + 2A_6;$
(123456)	$C_7 = 160A_0 + 32A_1 + 62A_2 + 40A_3 + 32A_4 + 25A_6 + 2A_7;$
(12)(3456)	$C_8 = 120A_0 + 64A_1 + 66A_2 + 28A_3 + 24A_4 + 18A_5 + 10A_6 + 4A_8;$
(12)(34)(56)	$C_9 = 140A_0 + 36A_1 + 10A_2 + 4A_3 + 8A_4 + 6A_5 + 8A_9;$
(123)(456)	$C_{10} = 160A_0 + 22A_2 + 32A_4 + 8A_5 + 5A_6 + 4A_{10}.$

TABLE III

P martix	1^6	$2, 1^4$	$3, 1^3$	$4, 1^2$	$2^2, 1^2$	$2, 3, 1$	$5, 1$	6	$2, 4$	2^3	3^2	μ_i
[12]	1	30	160	720	180	960	2304	3840	1440	120	640	1
[10, 2]	1	19	72	192	48	80	192	−384	−144	−12	−64	54
]8, 4]	1	12	16	−18	27	24	−144	−48	108	30	−8	275
[6, 6]	1	9	−8	−78	33	120	−48	−24	−114	−27	136	132
$[8, 2^2]$	1	9	22	12	−12	−60	−48	96	−24	−12	16	616
[6, 4, 2]	1	4	−8	−18	3	0	32	16	−4	−2	−24	2673
$[4^3]$	1	0	−20	30	15	−60	24	0	−60	30	40	462
$[6, 2^3]$	1	0	4	−6	−21	12	24	−48	12	6	16	1925
$[4^2, 2^2]$	1	−3	−8	24	3	0	−24	−12	24	−9	4	2640
$[4, 2^4]$	1	−8	12	−6	3	20	−24	48	−36	6	−16	1485
$[2^6]$	1	−15	40	−90	45	−120	144	−120	90	−15	40	132

Note. R. Roth has calculated these P matrices directly by computing the P_{ijk}'s, a monumental task, and the figures we have arrived at independently have each served as useful checks for the other.

APPLICATIONS. A lot of the results obtained by Delsarte [2] for Johnson and Hamming schemes carry over directly for the 1-factor schemes. Although this association scheme is not metric, it is possible to define a "generalized metric" on the 1-factors by saying that two 1-factors are at distance i if they have $n-i$ edges in common. One small difficulty arises in that there is no distance 1. If, however, d denotes the distance function, then $d(f_1,f_2)+d(f_2,f_3)\geqslant d(f_1,f_3)$ for any three 1-factors f_1, f_2, f_3.

One area for which these 1-factor schemes were developed was to see whether Delsarte's inequalities could be applied to the set of 99 1-factors of K_{12} which would arise from an oval of 12 points in a projective plane of order ten. Thompson [5] has considered this possibility in some detail and it was hoped that the inequalities would place some new restriction on the distribution vector of this set of 99 1-factors. The set of 99 forms a maximal clique with respect to relations 5–10. This is most easily seen by using the code clique theorem of Delsarte, and observing that the set of 105 1-factors containing 2 disjoint edges of K_{12} is a code with respect to the relations 5–10. The results however, were not very strong, and the only equalities arising could be arrived at more simply by other means.

Cameron [1] has asked for which relations between the 1-factors of K_{2n} is it possible to construct a 1-factorisation which consists of $(2n-1)$ 1-factors, any two of which are related in the same way. A 1-factorisation can again be considered as a maximal clique with respect to certain relations. Once more, the figures we have calculated only add a little to what was already known.

Although the applications of these association schemes seem rather limited at present, they are of some interest for their own sake, in particular, for the method used in calculating their eigenvalues, for the partial order which can be placed on both the A_i's and the E_i's, and for the generalised metric property they possess.

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